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A LIMIT THEOREM FOR ORDER PRESERVING NONEXPANSIVE OPERATORS IN L_1

BY

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ABSTRACT

We prove the following theorem: *Let T be an order preserving nonexpansive operator on $L_1(\mu)$ (or L_1^+) of a σ-finite measure, which also decreases the L_∞-norm, and let $S = tI + (1-t)T$ for $0 < t < 1$. Then for every $f \in L_p$ $(1 < p < \infty)$, the sequence $S^n f$ converges weakly in L_p.* (The assumptions do not imply that T is nonexpansive in L_p for any $p > 1$, even if μ is finite.) For the proof we show that $\| S^{n+1}f - S^n f \|_p \to 0$ for every $f \in L_p$, $1 < p < \infty$, and apply to S the following theorem: *Let T be order preserving and nonexpansive in L_1^+, and assume that T decreases the L_∞-norm. Then for $g \in L_p$ $(1 < p < \infty)$ $T^n g$ is weakly almost convergent. If for $f \in L_p$ we have $T^{n+1}f - T^n f \to 0$ weakly, then $T^n f$ converges weakly in L_p $(1 < p < \infty)$.*

1. Introduction

Order preserving operators T in L_1 (or L_1^+) of a σ-finite measure μ, which are integral preserving and satisfy $T0 = 0$, can serve as models for random motions of matter. Such operators arise also in some solutions of systems of partial differential equations [CT], and are necessarily nonexpansive in L_1, i.e., $\| Tf - Tg \|_1 \leq \| f - g \|_1$ for $f, g \in L_1$(or L_1^+) [CT], [KL$_1$].

The following ergodic theorem was proved in [KL$_1$]: "Let T be order preserving and nonexpansive in L_1 and assume that T decreases the L_∞-norm. Then for

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$f \in L_p$, $1 < p < \infty$, $A_n f = n^{-1} \sum_{k=0}^{n-1} T^k f$ converges weakly in L_p. If μ is finite, weak convergence in L_1 holds for $f \in L_1$."

An example in [LS] shows that the limit need not be a fixed point, even if $A_n f$ converges *strongly*. The example belongs to the class of disjointly additive operators studied in [KL$_2$]. The reason for that phenomenon seems to be that the set of fixed points is not convex, hence T is not nonexpansive in any L_p $(1 < p < \infty)$.

In this paper we study a commonly used iteration (introduced by Krasnoselskii [Kra] with $t = \frac{1}{2}$): Let $0 < t < 1$ and $S = tI + (1-t)T$, and iterate S. Clearly, S and T have the same fixed points, so strong convergence of S^n must be to a fixed point. Opial [O] proved that if T is nonexpansive in a bounded convex set of a Hilbert space, then $S^n x$ converges *weakly* to a fixed point. Strong convergence need not hold [GLind]. The first step in Opial's proof is the weak convergence of $T^n x$ (to a fixed point) when T is asymptotically regular ($\| T^{n+1}x - T^n x \| \to 0$ for every x). Bruck [B$_1$] showed that *weak* asymptotic regularity ($T^{n+1}x - T^n x \to 0$ weakly) is sufficient for the weak convergence of $\{T^n x\}$, by proving weak almost convergence in Baillon's ergodic theorem [Ba$_1$]. (The first application of almost convergence to nonlinear ergodic theory is due to Reich [Re$_2$].) Reich [Re] and Bruck [B$_2$] obtained weak almost convergence of $T^n x$ for nonexpansive maps with bounded orbits on closed convex sets in uniformly convex Banach spaces with Fréchet differentiable norm, thus extending Baillon's result [Ba$_2$]. Hirano [H] replaced the F-differentiability of the norm by Opial's original assumption. Ishikawa [I] proved (a general result implying) that $\| S^{n+1}x - S^n x \| \to 0$ for T nonexpansive with bounded orbits in any Banach space. This implies the (weak) convergence of $S^n x$ whenever (weak) almost convergence of $S^n x$ holds, a fact already used by Reich [Re].

In this paper we prove almost convergence of $T^n f$ (in the weak topology) for $f \in L_p$ $(1 < p < \infty)$, under the assumptions of [KL$_1$]. We show that, although T need *not* be nonexpansive in L_p $(1 < p < \infty)$, $\| S^{n+1} f - S^n f \|_p \to 0$ for $f \in L_p$, and therefore $S^n f$ converges weakly.

2. Limit theorems for order preserving nonexpansive operators in L_1

It is known in the linear case that ergodic theorems for a (linear) order preserving nonexpansive operator T on L_1 require that T reduce the L_∞-norm.

In the nonlinear case, norm reducing and nonexpansiveness are different notions and our operators need not be nonexpansive in L_p for $p > 1$.

LEMMA 2.1 [KL$_1$]. *If T is an order preserving nonexpansive operator in L_1 which decreases the L_∞-norm, then T is norm decreasing in L_p for $1 < p < \infty$.*

(A different proof is given in [LW], Lemma 2.2 and Proposition 2.4.)

Our first result is that in the mean ergodic theorem of [KL$_1$] we have almost convergence of $T^n f$ (in the weak topology of L_p, $1 < p < \infty$). Bruck [B$_1$] noted that Lorentz's theorem holds also in locally convex topological vector spaces: *x_n is almost convergent to x if and only if for every ergodic summation matrix $(a_{n,i})$, $\sum_{i=0}^{\infty} a_{n,i} x_i$ converges to x as $n \to \infty$.* (Recall that $(a_{n,i})$ is an *ergodic summation matrix* if $a_{n,i} \geq 0$, $\sum_{i=0}^{\infty} a_{n,i} = 1$ for every n, and $\lim_{n\to\infty} \sum_{i=0}^{\infty} |a_{n,i} - a_{n,i+1}| = 0$.)

In the weak topology, we need only Lorentz's original theorem.

For the proof of our result, along the lines of the proof of [KL$_1$], we need the following improvement of the quantative generalization [KL$_1$] of a lemma of Djafari-Rouhani. (This formulation allows us to obtain the weak almost convergence of the iterates of nonexpansive maps in Hilbert spaces.)

LEMMA 2.2. *Let (x_i) be a bounded sequence of vectors in a Hilbert space. Assume that, for some $\epsilon > 0$,*

$$\limsup_{i,j\to\infty} \sup_k [\|x_{i+k} - x_{j+k}\|^2 - \|x_i - x_j\|^2] \leq \epsilon. \tag{2.1}$$

Let $(a_{n,i})$ be an ergodic summation matrix, and put $y_n = \sum_{i=0}^{\infty} a_{n,i} x_i$. Then for any two weak limit points u and v of the sequence (y_n) we have $\|u - v\|^2 \leq \epsilon$.

PROOF: Let (m_k) and (n_k) be increasing sequences with $w-\lim y_{m_k} = u$, $w-\lim y_{n_k} = v$. By passing to subsequences and applying the diagonal argument, we may assume that the following limits exist:

$$\alpha = \lim_{k\to\infty} \sum_{i=0}^{\infty} a_{m_k,i} \|x_i\|^2,$$

$$\beta = \lim_{k\to\infty} \sum_{i=0}^{\infty} a_{n_k,i} \|x_i\|^2,$$

$$\varphi(j) = \lim_{k\to\infty} \sum_{i=0}^{\infty} a_{m_k,i} \|x_i - x_j\|^2,$$

$$\psi(j) = \lim_{k\to\infty} \sum_{i=0}^{\infty} a_{n_k,i} \|x_i - x_j\|^2.$$

By taking further subsequences, we may assume the existence of

$$\varphi_\alpha = \lim \sum_{j=0}^{\infty} a_{m_k,j} \varphi(j), \qquad \varphi_\beta = \lim \sum_{j=0}^{\infty} a_{n_k,j} \varphi(j),$$

$$\psi_\alpha = \lim \sum_{j=0}^{\infty} a_{m_k,j} \psi(j), \qquad \psi_\beta = \lim \sum_{j=0}^{\infty} a_{n_k,j} \psi(j).$$

For $\epsilon' > 0$, (2.1) implies that there exists I such that for $i,j \geq I$, $k \geq 0$, $\|x_{i+k} - x_{j+k}\|^2 \leq \|x_i - x_j\|^2 + \epsilon + \epsilon'$.

By the "shift invariance" of the rows in $(a_{n,i})$ (and boundedness of (x_n)), we obtain

$$\varphi(j + k) \leq \varphi(j) + \epsilon + \epsilon' \qquad \text{for } j \geq I.$$

Hence $\limsup_{j\to\infty}\varphi(j) \le \liminf_{j\to\infty}\varphi(j) + \epsilon$, and thus, using again the "shift invariance", we obtain $|\varphi_\alpha - \varphi_\beta| \le \epsilon$. Similarly, $|\psi_\alpha - \psi_\beta| \le \epsilon$. Now

$$\begin{aligned}\langle y_n, y_m\rangle &= \textstyle\sum_i \sum_j a_{m,i}a_{n,j}\langle x_i, x_j\rangle \\ &= \tfrac{1}{2}\textstyle\sum_i\sum_j a_{m,i}a_{n,j}\|x_i\|^2 + \tfrac{1}{2}\sum_i\sum_j a_{m,i}a_{n,j}\|x_j\|^2 \\ &\quad - \tfrac{1}{2}\textstyle\sum_i\sum_j a_{m,i}a_{n,j}\|x_i - x_j\|^2 \\ &= \tfrac{1}{2}\textstyle\sum_i a_{m,i}\|x_i\|^2 + \tfrac{1}{2}\sum_j a_{n,j}\|x_j\|^2 - \tfrac{1}{2}\sum_i\sum_j a_{m,i}a_{n,j}\|x_i - x_j\|^2.\end{aligned}$$

Put $m = m_k$, and let $k \to \infty$ to obtain

$$\langle y_n, u\rangle = \frac{1}{2}\left[\alpha + \sum_j a_{n,j}\|x_j\|^2 - \sum_j a_{n,j}\varphi(j)\right].$$

Putting $n = m_k$ and letting $k \to \infty$, we get $\langle u,u\rangle = (\alpha + \alpha - \varphi_\alpha)/2$. Putting $n = n_k$ and letting $k \to \infty$, we get $\langle v,u\rangle = (\alpha + \beta - \varphi_\beta)/2$. Similarly, putting first $n = n_k$ in the expression for $\langle y_n, y_m\rangle$ and letting $k \to \infty$, and then $m = n_k$ or $m = m_k$, we have $\langle v,v\rangle = (\beta + \beta - \psi_\beta)/2$ and $\langle u,v\rangle = (\alpha + \beta - \psi_\alpha)/2$. Hence

$$\|u - v\|^2 = \langle u,u\rangle + \langle v,v\rangle - 2\langle u,v\rangle = (-\varphi_\alpha - \psi_\beta + \varphi_\beta + \psi_\alpha)/2 \le \epsilon.$$

THEOREM 2.3. *Let T be order preserving and nonexpansive in $L_1(\mu)$ (or in L_1^+) and assume that T decreases the L_∞-norm. Then for any $f \in L_p$, $1 < p < \infty$, $T^n f$ is almost convergent in the weak topology, i.e., for any ergodic summation matrix $(a_{n,i})$, $\sum_{i=0}^\infty a_{n,i}T^i f$ converges weakly. If μ is finite, weak almost convergence in L_1 holds for any $f \in L_1$.*

PROOF. We follow the proof of [KL₁], by noting the changes.

First reduction. It is enough to prove the theorem (when T is defined on all of L_1) only for T on L_1^+. The proof of Lemma 4.2 in [KL₁] gives the estimate on the individual terms, and the "shift invariance" of the rows in $(a_{n,i})$ yields the result.

We fix a given $(a_{n,i})$, and define $A_n f = \sum_{i=0}^\infty a_{n,i}T^i f$. For $f \in L_p$, $\|T^i f\|_p \le \|f\|_p$ by Lemma 2.1, so also $\|A_n f\|_p \le \|f\|_p$.

Second reduction. It is enough to prove for $0 \le f \in L_p$ bounded: The proof in [KL₁] applies verbatim with the extended definition of $A_n f$.

Third reduction. It is enough to prove for $f \in L_p \cap L_\infty^+$ integrable: The estimate in [KL₁] is for individual terms, so the proof there applies, writing, at the end of that proof, $A_n(S) = \sum_{j=0}^\infty a_{nj}S^j$.

PROOF OF THE THEOREM. Since Lemmas 4.4 and 4.5 of [KL$_1$] remain unchanged, in the proof of [KL$_1$] we have only to redefine $\alpha_{m,n} = \sum_{i=0}^{\infty} a_{n,i} x_{m,i}$. Since $T^i f = \sum_{m=0}^{M} x_{m,i}$, we still have $\sum_{m=0}^{M} \alpha_{m,n} = \sum_{i=0}^{\infty} a_{n,i} T^i f = A_n f$, and the proof remains the same as in [KL$_1$], using our Lemma 2.2 instead of Lemma 4.3 there.

Finally, note that if for every ergodic summation matrix we have convergence, then the limit is the same for all matrices (otherwise we intertwine the rows of two matrices to obtain a new one with two limit points).

REMARK. The almost convergence is not just a slight generalization of the ergodic theorem, but is crucial for the next result.

COROLLARY 2.4. *Let T satisfy the hypotheses of Theorem 2.3. If $T^n f - T^{n+1} f \to 0$ weakly in L_p $(1 < p < \infty)$ for a given $f \in L_p$, then $T^n f$ converges weakly.*

PROOF. $T^n f$ is almost convergent weakly, by Theorem 2.3, so $T^{n+1} f - T^n f \to 0$ weakly implies convergence, by [B$_1$].

THEOREM 2.5. *Let T be order preserving and nonexpansive in $L_1(\mu)$ (or in L_1^+) and assume that T decreases the L_∞-norm. For fixed $0 < t < 1$ define $Sf = tf + (1-t)Tf$. Then for $f \in L_p$, $1 < p < \infty$, $S^n f$ converges weakly. If μ is finite, $S^n f$ converges weakly in L_1 for any $f \in L_1$.*

PROOF. By Lemma 2.1, T is norm decreasing in L_p, $1 < p < \infty$. Theorem 2.5 follows from Corollary 2.4 and the following general lemma.

LEMMA 2.6. *Let K be a convex subset of a uniformly convex Banach space, and let T be a norm-decreasing map (i.e., $\|Tx\| \le \|x\|$) of K into itself. Then the operator $Sx = tx + (1-t)Tx$, defined for $0 < t < 1$, satisfies $\|S^{n+1}x - S^n x\| \to 0$ for every $x \in K$.*

PROOF. Fix $x \in K$. By our assumption, $\|S^n x\|$ is decreasing (since S is also norm-decreasing), so let $D = \lim \|S^n x\| = \inf \|S^n x\|$. We have to prove the lemma only for $D > 0$.

Fix $\epsilon > 0$. By uniform convexity, there exists $\delta > 0$ such that whenever $\|y\|, \|z\| \le D + \delta$ and $\|tz + (1-t)y\| \ge D$, then $\|y - z\| \le \epsilon$. Let $\|S^N x\| < D + \delta$. Then, for $n > N$,

$$\|TS^n x\| \le \|S^n x\| \le \|S^N x\| < D + \delta,$$

$$\|tS^n x + (1-t)TS^n x\| = \|S^{n+1} x\| \ge D.$$

Hence $\|S^n x - TS^n x\| \le \epsilon$ and $\|S^n x - S^{n+1} x\| \le \epsilon$ for $n > N$.

REMARKS. (1) *Pointwise* convergence of $S^n f$ need not hold under the assumptions of Theorem 2.5, even if μ is finite and T is *linear* (hence nonexpansive in L_∞). J. Rosenblatt [R] showed that for every invertible ergodic measure preserving transformation τ on an atomless probability space, there exists a measurable set A such that $\{S^n 1_A\}$ is not a.e. convergent ($Tf = f \circ \tau$, of course).

(2) We may use Ishikawa's result [I] for the proof of Theorem 2.5: For T in L_1 we obtain $\|S^{n+1}f - S^n f\|_1 \to 0$ for $f \in L_1$, hence $S^{n+1}f - S^n f \to 0$ weakly in L_p ($1 < p < \infty$), with strong L_p-convergence when μ is *finite*. However, Lemma 2.6 has a much simpler proof, and yields additional information when μ is infinite.

(3) If in Lemma 2.6 the space is a Hilbert space and $t = \frac{1}{2}$, we have even

$$\sum_{n=0}^{\infty} \|S^n x - S^{n+1}x\|^2 \leq (\|x\|^2 - \lim \|S^n x\|^2).$$

PROOF. By the parallelogram identity,

$$\|x + Tx\|^2 + \|x - Tx\|^2 \leq 4\|x\|^2.$$

Hence

$$\|x - Tx\|^2 \leq 4(\|x\|^2 - \|Sx\|^2),$$

so using $Sx = \frac{1}{2}(x + Tx)$ we have

$$\|x - Sx\|^2 = \frac{1}{4}\|x - Tx\|^2 \leq \|x\|^2 - \|Sx\|^2.$$

We now use this for $S^n x$ instead of x and sum over n.

(Note: $\sum \|S^n x - S^{n+1}x\|$ need not converge, by [GLind].)

We now want to exhibit a class of examples where T satisfies the assumptions of Theorem 2.5, but T is *not* nonexpansive in L_p for $p > 1$. We will obtain from Proposition 2.9 below that any T which is order preserving, integral preserving and disjointly additive on L_1 (or L_1^+) of a finite measure space is *not* nonexpansive in L_p for $p > 1$, unless it is linear. A construction of nonlinear examples, with T also L_∞ norm decreasing, was given in [KL$_1$], and further studied in [KL$_2$]. The example in [LS], where $N^{-1}\sum_{n=0}^{N-1} T^n f$ converges strongly to a non-invariant limit, also belongs to that class.

For the sake of completeness, we include the proof from [LW] of the following lemma:

LEMMA 2.7. *Let T be order preserving and integral preserving on L_1 (L_1^+) of a probability space. Assume that, for some $1 < p \leq \infty$, T is nonexpansive in L_p. Then for any $f \in L_1$ and t ($t \geq 0$) we have $T(f + t) = Tf + t$.*

PROOF. Note that T is nonexpansive on L_1 by [KL$_1$].

Since $\mu(\Omega) = 1$, for $f \in L_p$ and $t > 0$ we have

$$t = \int [(f+t) - f]\,d\mu = \int [T(f+t) - Tf]\,d\mu \le \|T(f+t) - Tf\|_p \|1\|_q \le t,$$

using preservation of integrals and Hölder's inequality. We obtain equality in Hölder's inequality, so $T(f+t) - Tf$ must be a constant, which is t since T preserves integrals. Now $T(f+t) = Tf + t$ for $f \in L_1$ and $t \ge 0$ follows by continuity. When T is defined on all of L_1, and $t < 0$, then $Tf = T(f+t-t) = T(f+t) - t$.

COROLLARY 2.8. *The assumptions of Lemma* 2.7 *with* $1 < p < \infty$ *imply that* T *is nonexpansive in* L_∞.

REMARK. If we assume only that T is norm decreasing in L_p, it is also in L_∞.

DEFINITION [KL$_2$]. T is called *disjointly additive* if $fg = 0 \Rightarrow T(f+g) = Tf + Tg$.

PROPOSITION 2.9. *Let T be disjointly additive on L_1 (L_1^+) of a probability space, order preserving and integral preserving. If, for some $1 < p < \infty$, T is nonexpansive in L_p, then T is linear.*

PROOF. The case of L_1^+ is simpler (some parts of the following proof should be omitted), so we prove the L_1 case. By disjoint additivity, $T0 = 0$.

Let A be measurable, $\alpha, \beta \ge 0$. Using disjoint additivity and applying Lemma 2.7 to $f = 1_A$ we obtain (since $Tc = c$ from Lemma 2.7 with $f = 0$)

$$\begin{aligned} T(\alpha + \beta 1_A) &= \alpha + T(\beta 1_A) = T(\alpha 1_A + \alpha 1_{A^c}) + T(\beta 1_A) \\ &= T(\alpha 1_A) + T(\alpha 1_{A^c}) + T(\beta 1_A). \end{aligned}$$

From disjoint additivity we get

$$T(\alpha + \beta 1_A) = T(\alpha 1_{A^c} + (\alpha + \beta) 1_A) = T(\alpha 1_{A^c}) + T((\alpha + \beta) 1_A).$$

Hence $T((\alpha+\beta)1_A) = T(\alpha 1_A) + T(\beta 1_A)$. By induction, $T(n\alpha 1_A) = nT(\alpha 1_A)$ for n a natural number. Replacing α by α/n we get

$$T\left(\frac{1}{n}\alpha 1_A\right) = \frac{1}{n} T(\alpha 1_A),$$

and then conclude ($m\alpha$ instead of α) that $T(t\alpha 1_A) = tT(\alpha 1_A)$ for $t > 0$ rational, and, by continuity, for $t > 0$ real. For $\alpha > 0$, $T\alpha = \alpha$, by Lemma 2.7, and we get $T(\alpha 1_{A^c}) = \alpha + T(-\alpha 1_A)$. Hence $T(-t\alpha 1_A) = -T(t\alpha 1_A) = -tT(\alpha 1_A)$ for $t > 0$.

Thus $T(t\alpha 1_A) = tT(\alpha 1_A)$ for any t real, $\alpha \geq 0$. We use $T(-\alpha 1_A) = -T(\alpha 1_A)$ again to obtain $T(t\alpha 1_A) = tT(\alpha 1_A)$ for any t and α. Together with disjoint additivity, T is homogeneous on simple functions, hence T is homogeneous on $L_1(\mu)$ by continuity. Together with disjoint additivity, we obtain that T is additive on simple functions, and, by continuity, T is additive.

3. On convergence in Hilbert spaces and in L_p

We have not been able to decide if (or when) the convergence in Theorem 2.5 is in the L_p norm. We collect in this section some results related to the lemmas in the previous section, which deal with strong convergence.

LEMMA 3.1. *Let (x_n) be a sequence of vectors in a uniformly convex Banach space, satisfying*:

(i) $\|x_{n+1} + x_{m+1}\| \leq \|x_n + x_m\|$ *for* $n, m \geq 0$,

(ii) $\liminf_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Then x_n converges strongly.

PROOF. By (i) with $m = n$, $\|x_n\|$ is decreasing, so we have to prove only when $\lim\|x_n\| = \inf\|x_n\| = D$ is positive. Fix $\epsilon > 0$. By the uniform convexity, there exists $\delta > 0$ such that for y, z satisfying $\|y\|, \|z\| \leq D + \delta$ and $\|y + z\| \geq 2D - \delta$ we must have $\|y - z\| \leq \epsilon$. Let m and k satisfy

(1) $\liminf_{n\to\infty} \|x_{n+k} - x_{n+m}\| = 0$.

By (ii), (1) holds for $k = m + 1$, $m \geq 0$.

Fix N such that $\|x_N\| \leq D + \delta$. Take $n > 0$ such that $n + k > N$, $n + m > N$. We can find $n' > n$ with $\|x_{n'+k} - x_{n'+m}\| \leq \delta$, by (1). Hence, by (i), we have

$$\|x_{k+n} + x_{m+n}\| \geq \|x_{k+n'} + x_{m+n'}\| \geq 2\|x_{k+n'}\| - \delta \geq 2D - \delta.$$

Since $\|x_{k+n}\| \leq D + \delta$ and $\|x_{m+n}\| \leq D + \delta$, we have by the definition of δ that $\|x_{m+n} - x_{k+n}\| \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\lim_{n\to\infty} \|x_{n+k} - x_{n+m}\| = 0$ for k and m satisfying (1). Thus $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ by (ii). Hence any pair (m, k) satisfies (1).

To show that (x_n) is a Cauchy sequence, note that, for $\epsilon > 0$, if $m, k > N$ we can take $n = 0$ in the above argument to get $\|x_m - x_k\| \leq \epsilon$.

REMARK. In Hilbert spaces, condition (i) of Lemma 3.1 is already sufficient to ensure strong convergence of $(1/N) \sum_{n=1}^{N} x_n$; see [W]. In fact, strong almost convergence holds.

Our next lemma gives in Hilbert spaces quantitative estimates related to the previous lemma, in the spirit of Lemma 2.2.

LEMMA 3.2. *Let $\{x_n\}$ be a sequence of vectors in a Hilbert space which satisfies*:

(i) $\lim_n \|x_n\| = D$ *exists.*

(ii) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

(iii) $\limsup_{n,m\to\infty} \liminf_{k\to\infty} [\|x_{n+k} + x_{m+k}\|^2 - \|x_n + x_m\|^2] \le \epsilon$ $(\epsilon \ge 0)$.

Then $\limsup_{n,m\to\infty} \|x_n - x_m\|^2 \le \epsilon$.

PROOF. If $D = 0$ the result is obvious. For $D \ne 0$ we may, and do, assume $D = 1$.

Let $\delta > 0$. By (i), there exists N such that $1 - \delta < \|x_n\| < 1 + \delta$, for $n > N$. By (ii), for each fixed j we have $\|x_{k+j} - x_k\| \xrightarrow[k\to\infty]{} 0$. By (iii), there exists M such that for fixed $m, n > M$ we have $\|x_{n+k} + x_{m+k}\|^2 - \|x_n + x_m\|^2 \le \epsilon + \delta$ for infinitely many k.

Fix $m, n > M \vee N$. For k large enough $\|x_{m+k} - x_{n+k}\| < \delta$, since $\lim_k \|x_{m+k} - x_{n+k}\| = \lim_k \|x_{(m-n)+k} - x_k\| = 0$. The large k is chosen so that (consequence of (iii)) $\|x_{n+k} + x_{m+k}\|^2 \le \|x_n + x_m\|^2 + \epsilon + \delta$. Clearly, also $1 - \delta < \|x_{n+k}\| < 1 + \delta$, $1 - \delta < \|x_{m+k}\| < 1 + \delta$, so

$$\|x_{m+k} + x_{n+k}\| = \|2x_{n+k} + (x_{m+k} - x_{n+k})\| \ge 2\|x_{n+k}\| - \delta \ge 2 - 3\delta.$$

Hence

$$\begin{aligned}\|x_m - x_n\|^2 + (2 - 3\delta)^2 - \delta - \epsilon &\le \|x_m - x_n\|^2 + \|x_{m+k} + x_{n+k}\|^2 - \delta - \epsilon \\ &\le \|x_m - x_n\|^2 + \|x_m + x_n\|^2 = 2(\|x_m\|^2 + \|x_n\|^2) \le 4(1 + \delta)^2.\end{aligned}$$

Thus, $\|x_m - x_n\|^2 \le \epsilon + 21\delta$. Hence $\limsup_{n,m\to\infty} \|x_m - x_n\|^2 \le \epsilon + 21\delta$ for every $\delta > 0$, so the result follows.

REMARKS. (1) Putting $\epsilon = 0$ we obtain a convergence theorem for x_n. Of course, if x_n converges, all three conditions of Lemma 3.2 are *necessary* (with $\epsilon = 0$).

(2) Condition (i) implies (by taking $m = n$ large) that the left-hand side of (iii) is non-negative.

The following lemma, trivial when μ is finite, shows that if we assume in Theorem 2.5 that T is also nonexpansive in L_∞, then L_p-strong convergence of $S^n f$ for every $f \in L_p$, $1 < p < \infty$, will follow from L_2-strong convergence for every $f \in L_2$. (Nonexpansiveness in L_p is needed for the approximation.)

LEMMA 3.3. *Let $\{f_i\}$ be a sequence in $L_1 \cap L_\infty$ with $\|f_i\|_1 \le K_1$, $\|f_i\|_\infty \le K_\infty$. If $\{f_i\}$ is a Cauchy sequence in L_2, then it is a Cauchy sequence in every L_p, $1 < p < \infty$.*

PROOF. Since $\|f_i\|_p^p = \int |f_i|^p d\mu \leq \|f_i\|_1 \|f_i\|_\infty^{p-1}$, there exists $K_p > 0$ such that $\|f_i\|_p \leq K_p$ for every i.

The lemma follows from the following assertion:

For every n, $\{f_i\}$ is a Cauchy sequence in L_p if $p \geq \dfrac{n+1}{n}$.

For $n = 1$, if $p \geq 2$ then

$$\|f_i - f_j\|_p^p = \int |f_i - f_j|^p d\mu = \int |f_i - f_j|^2 |f_i - f_j|^{p-2} d\mu \leq \|f_i - f_j\|_2^2 (2K_\infty)^{p-2},$$

and $\{f_i\}$ is a Cauchy sequence in L_p, since it is in L_2 by assumption.

Assume the assertion to be true for $n = k$, and let $p \geq (k+2)/(k+1)$. Then $(p-1)(k+1) \geq 1$, and thus

$$\| |f_i - f_j|^{p-1} \|_{k+1}^{k+1} = \int |f_i - f_j|^{(p-1)(k+1)} d\mu$$

$$= \|f_i - f_j\|_{(p-1)(k+1)}^{(p-1)(k+1)} \leq (2K_{(p-1)(k+1)})^{(p-1)(k+1)}.$$

Hence, by Hölder's inequality,

$$\|f_i - f_j\|_p^p = \int |f_i - f_j| \, |f_i - f_j|^{(p-1)} d\mu \leq \|f_i - f_j\|_{(k+1)/k} \| |f_i - f_j|^{p-1} \|_{k+1}$$

$$\leq \|f_i - f_j\|_{(k+1)/k} (2K_{(p-1)(k+1)})^{p-1}.$$

It follows from the induction hypothesis that $\{f_i\}$ is a Cauchy sequence in L_p, $p \geq (k+2)/(k+1)$.

REMARKS. (1) If μ is finite, $\{f_i\}$ is clearly a Cauchy sequence in $L_1(\mu)$, and then in any L_p, $1 < p < \infty$. Thus, Lemma 3.3 is non-trivial only for μ infinite.

(2) We may assume in the lemma that $\{f_i\}$ is a Cauchy sequence in L_{p_0}, for some $1 < p_0 < \infty$, since $\|f_i - f_j\|_2^2 \leq \|f_i - f_j\|_{p_0} \|f_i - f_j\|_{p_0/(p_0-1)}$ shows that the conclusion holds.

REFERENCES

[B$_1$] R. E. Bruck, *On the almost convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak* ω-limit set, Isr. J. Math. **29** (1978), 1–16.

[B$_2$] R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Isr. J. Math. **32** (1979), 107–116.

[Ba$_1$] J. B. Baillon, *Un théorème de type ergodique pour les contractions non-linéaires dans un espace de Hilbert*, C.R. Acad. Sci. Paris **A280** (1975), 1511–1514.

[Ba_2] J. B. Baillon, *Comportement asymptotique des itérés de contractions non-linéaires dans les espaces L_p*, C.R. Acad. Sci. Paris **A286** (1978), 157-159.

[CT] M. G. Crandall and L. Tartar, *Some relations between nonexpansive and order preserving mappings*, Proc. Am. Math. Soc. **78** (1980), 385-390.

[GLind] A. Genel and J. Lindenstrauss, *An example concerning fixed points*, Isr. J. Math. **22** (1975), 81-86.

[H] N. Hirano, *Nonlinear ergodic theorems and weak convergence theorems*, J. Math. Soc. Japan **34** (1982), 35-46.

[I] S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Am. Math. Soc. **59** (1976), 65-71.

[Kra] N. Krasnoselskii, *Two remarks on the method of successive approximations*, Uspehi Mat. Nauk **10** (1955), 123-127.

[KL_1] U. Krengel and M. Lin, *Order preserving nonexpansive operators in L_1*, Isr. J. Math. **58** (1987), 170-192.

[KL_2] U. Krengel and M. Lin, *An integral representation of disjointly additive order preserving operators in L_1*, Stochastic Analysis and Appl. **6** (1988), 289-304.

[LS] M. Lin and R. Sine, *On the fixed point set of nonexpansive order preserving maps*, Math. Z. **203** (1990), 227-234.

[LW] M. Lin and R. Wittmann, *Pointwise ergodic theorems for certain order preserving mappings in L_1*, preprint.

[O] Z. Opial, *Weak convergence of successive approximations for nonexpansive mappings in Banach spaces*, Bull. Am. Math. Soc. **73** (1967), 591-597.

[R] J. Rosenblatt, *Ergodic group actions*, Arch. Math. **47** (1986), 263-269.

[Re] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274-276.

[Re_2] S. Reich, *Almost convergence and nonlinear ergodic theorems*, J. Approx. Theory **24** (1977), 269-272.

[W] R. Wittmann, *Mean ergodic theorems for nonlinear operators*, Proc. Am. Math. Soc. **108** (1990), 781-788.